

Fractal spin structures as origin of $1/f$ magnetic noise in superconducting circuits

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We analyze recent data on the complex inductance of dc SQUIDs that show $1/f$ inductance noise highly correlated with conventional $1/f$ flux noise. We argue that these data imply a formation of long range order in fractal spin structures. We show that these structures appear naturally in a random system of spins with wide distribution of spin-spin interactions. We perform numerical simulations on the simplest model of this type and show that it exhibits $1/f^{1+\zeta}$ magnetization noise with small exponent ζ and reproduces the correlated behavior observed experimentally.

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Despite recent progress, the origin of low-temperature ($T \lesssim 1K$) $1/f$ flux noise in superconducting circuits remains an open question for nearly 30 years [1]. A progress in this problem would be important for a number of applications: noise suppression would improve the sensitivity of SQUID magnetometers [1, 2] and eliminate the dominant source of decoherence for flux [3] and phase [4] qubits. The problem has two parts: the microscopic origin of the degrees of freedom that produce the noise and the origin of the interaction responsible for their dynamics.

It is now accepted that the noise is due to the electron spins localized at the surfaces of the superconductors [4–6]. The surface localization is implied by the weak dependence of the noise amplitude on the area of SQUIDs: variation of the loop area over 5 orders of magnitudes does not lead to a significant systematic change in the noise amplitude [1, 5]. The approximate temperature independence of the noise in the range $20mK \lesssim T \lesssim 500mK$ suggests that the magnetic system responsible for the noise must be characterized by very low energy scales pointing to electron or nuclear spins. The nuclear spins can be ruled out because $1/f$ flux noise persists up to 10MHz [7] which is much larger than the typical energy scale associated with nuclear spins $f \lesssim 1kHz$ [6]. Recent data [8, 9] confirm that the flux noise is due to electron spins situated close to the surface of superconductors and show that these spins are characterized by the surface density $n_s \sim 5 \times 10^{17}m^{-2}$. Further support for this conclusion is provided by density functional calculations [10] which found that localized electronic levels are likely to occur at strongly disordered metal-insulator (MI) interfaces.

The dynamics of the electron spins that generate the noise remains poorly understood. However, very recently, an experimental work by S. Sendelbach et al. [11] discovered a highly unusual feature of the magnetic noise in dc-SQUIDs at millikelvin temperatures that severely limits possible mechanisms. It was found that the noise in the SQUID inductance has also $1/f$ power spectrum and that the inductance fluctuations are highly correlated with the usual $1/f$ flux noise. Moreover, the correlation between the two noises grows as the temperature is decreased and

becomes of the order of unity below $T \sim 100mK$. This suggests a common underlying mechanism producing the noise in both inductance and flux. Because inductance and flux are even and odd under time inversion operation respectively their cross-correlation implies broken time inversion symmetry and the appearance of a long range magnetic order.

The formation of magnetic order is difficult to reconcile with the temperature independence of the flux noise because normally the former implies local fields of the order of the transition temperature, T_c , that suppress individual spin dynamics at $T \ll T_c$. The collective modes are expected to be suppressed by anisotropy at the temperatures in mK range so they cannot lead to temperature independent flux noise. Similarly, the transitions between different metastable states in ordered magnets involve domain wall motion which disappears at low temperatures. Finally, the spin glass formation has to be ruled out because one does not expect correlations between magnetization and susceptibility in spin glasses, this expectation was confirmed by simulations [12].

In this paper we show that the experimental results [11] are reproduced in a highly disordered spin model where interaction between spins is characterized by a very broad distribution of the coupling strengths. In this situation the long range order is due to formation of a large fractal cluster of strongly coupled spins which spans the whole system. This ordered cluster includes only a fraction of spins leaving many smaller clusters and isolated spins free to fluctuate. We formulate the simplest model of this type and provide numerical results that agree with the existing data. In more detail, we consider a system of Ising spins distributed randomly in a 2D plane. Spin-spin interactions are assumed ferromagnetic and decaying exponentially with distance which provides a broad distribution of couplings between them. We find that magnetization dynamics simulated by single spin flip Monte Carlo algorithm produces noise with $1/f$ power spectra. We computed the linear response, $\chi(t)$, of the system to a magnetic field of the low frequency, ω , and fluctuations of the magnetization of the system, $M(t)$, at time scales much longer than the period of the external field,

$t\omega \gg 1$. These quantities mimic the properties measured in the experiments [11, 13]. Our main result is a large cross-correlation between the noise in the susceptibility and magnetization, as illustrated by Fig. 1, in a wide range of temperatures in the ferromagnetic phase.

The model provides a new insight into the nature of spin-spin interactions which is the most likely mechanism generating the dynamics of the spins. First, we note that both phonons and nuclear spins can be ruled out as a source of electron spin dynamics due to their weak coupling to the electron spins [5, 6]. Alternative sources of the dynamics such as hopping of electrons between traps with random spin orientations [5] or interaction with tunneling two level systems (TLS) [14] are difficult to reconcile with experimental data. The former requires very high concentration of single-electron excitations while the latter implies high concentration of thermally activated TLSs [6]. This leaves only electron spin interaction with each other.

The typical spin-spin distance inferred from the reported densities is $\gtrsim 1nm$, which implies an exponentially small direct spin exchange. However, proximity to the metal makes RKKY interaction possible [6]. This interaction can be especially large for the electrons localized on metal-insulator boundary levels that remain singly occupied due to Coulomb repulsion [10], while electrons localized deeper in the insulator are expected to have exponentially weaker coupling to the conduction electrons. The exponential dependence of the coupling to conduction electrons implies a broad distribution of spin-spin couplings.

Theoretical analysis of a system of spins interacting via such random RKKY mechanism under assumption of *sufficiently weak* interactions (such that the system does not

form an ordered state) showed that the system can generate temperature independent $1/f$ noise. Relatively high frequency part of the noise spectra [2] is generated by the diffusion of local magnetization [6], whereas the very low frequency part [4, 15] is generated by fluctuations of rare pairs of spins [16].

Temperature independence of the $1/f$ noise and correlations between flux and inductance [11] imply unusual magnetic long range order. In a conventional magnet the noise is due to domain wall jumps. Each such jump induces changes in the magnetization and susceptibility proportional to the changes in the domain volume δV and domain wall area δS respectively. This means that in conventional magnets where $S \ll V$ the fluctuations in the response to an external field is much smaller than fluctuations in the quasistatic magnetization in contrast to the data.

Large fluctuations of the response and as well as high correlations between the response and magnetization are possible in highly disordered magnets with wide distributions of spin-spin couplings where correlated spins form fractal clusters with $V \sim S$. The simplest model that contains this physics is characterized by the Hamiltonian

$$\mathcal{H} = - \sum_{ij} J_{ij} s_i s_j, \quad J_{ij} = J_0 \exp(-r_{ij}/a), \quad (1)$$

where i numbers sites of the random lattice, a is a decay length of interactions, J_0 determines the annealing temperature of the system and spins are classical $s = \pm 1$. Three dimensional version of this model was used earlier to analyze the formation of long range order in magnetically doped semiconductors [17, 18].

At small densities $na^2 \ll 1$ the system (1) undergoes finite temperature ferromagnetic transition which is driven either by the temperature T or by the lattice site density n . This transition is in the universality class of percolation transition. It can be understood in terms of the "circle packing" problem in 2D in which one considers circles of radius, $r(T)/2 \equiv a/2 \ln J_0/T$, drawn around each spin site. Overlapping circles correspond to spins separated by $r_{ij} \lesssim r(T)$ which are therefore strongly interacting and, thus, aligned. In contrast, spins separated by $r_{ij} > r(T)$ are effectively independent. Dimensionless parameter $B(T) = \pi r^2(T)n$ controls the thermodynamic transition. At high temperatures $B(T) \ll 1$ strongly coupled spins are rare and form only small clusters. As the temperature decreases, $B(T)$ increases, both the number and sizes of the clusters grow and at some threshold value $B = B_c$ an infinite cluster of strongly coupled spins forms. This corresponds to the ferromagnetic ordering at the critical temperature, $T_c = J_0 \exp\left(-\sqrt{B_c}/(\pi na^2)\right)$. The threshold value for the "circle packing" problem is $B_c \approx 4.5$ [19]. In the vicinity of the percolating transition the cluster size distribution is given by the power law: $n(s) \sim s^{-\tau} \exp(-Cs^{1/2})$, $B > B_c$, where C is a constant and τ is a characteristic exponent [19]. The scaling behavior is realized for $|B - B_c| \ll B_c$ which translates

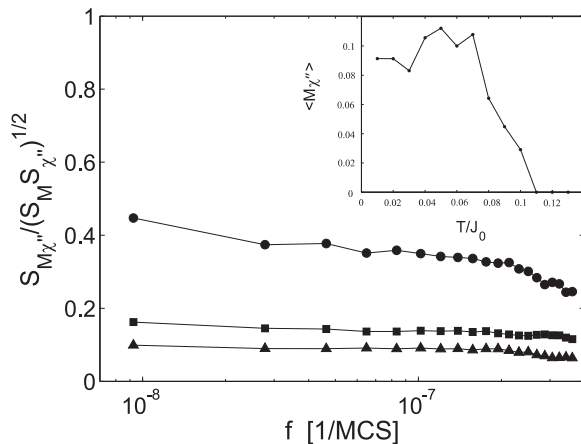


Figure 1: Cross correlator of noise power spectra of the magnetization and susceptibility for $T = 0.05J_0$ and a particular disorder configuration (circles); $T = 0.05J_0$ averaged over disorder (squares); $T = 0.1J_0$ averaged over disorder (triangles). Inset shows temperature dependence of the normalized disorder averaged correlator of magnetization and susceptibility fluctuations $\langle M\chi'' \rangle$.

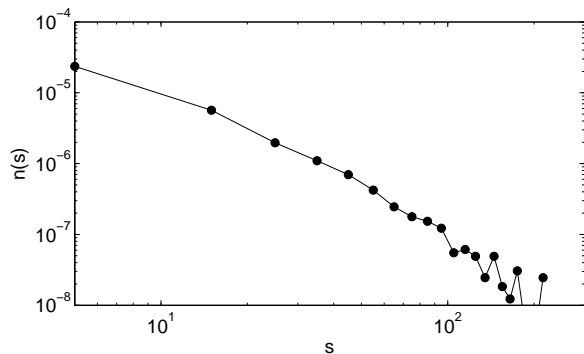


Figure 2: Cluster size distribution $n(s)$ at $|B_c - B|/B_c = 0.2$. Up to sizes $s \gtrsim 100$ the data fits a power law $n(s) \sim s^{-\tau}$ with $\tau \sim 2$.

into a wide range of temperatures: $\ln(T_c/T) \ll 1/\sqrt{na^2}$.

We analyzed numerically the cluster size distribution in the model (1) in the parameter range $0 \lesssim |B_c - B|/B_c \lesssim 0.2$. This translates into $|\ln T_c/T| \lesssim 0.5$ for the model characterized by realistic values of $\ln J_0/T_c \approx 5$. We found a broad power law distribution of cluster sizes up to $s \gtrsim 100$ at $|B_c - B|/B_c = 0.2$, see Fig. 2, indicating that in this whole temperature range the spin system separates into many large clusters weakly coupled to each other and to the infinite percolation cluster.

Formation of large clusters of spins that flip as whole objects under thermal excitation affect the magnetization dynamics of the system for two reasons. First, the contribution to the noise coming from large clusters scales as $n(s)s^2$ so their relatively small number is compensated by the additional factor of s . Second, the broad distribution of the cluster sizes translates into wide distribution of relaxation rates Γ associated with cluster dynamics which results in $1/f$ power spectra of the magnetization noise.

In order to check this conjecture we have simulated the dynamics of the spin system (1) in the critical regime using single spin flip Monte Carlo dynamics satisfying the detailed balance condition. We focus on the fluctuations of magnetization in the regime below the freezing temperature, T_c , at which the infinite cluster is formed. To minimize the transient regime we choose for the initial configuration that has spins on the infinite cluster aligned in one direction whereas all others are distributed randomly. Very long runs up to 3×10^{10} Monte Carlo steps (MCS) were performed to analyze long time dynamics in the steady state characterized by finite average magnetization. We observe $1/f^\alpha$ power spectra of the noise in magnetization of the system, Fig 3, with the noise exponent $0.8 \lesssim \alpha \lesssim 1.2$.

In order to simulate the susceptibility noise measurement [11], we subjected the spin system to a time-dependent magnetic field $H = H_0 \cos 2\pi\omega t$ and simulated the lock-in detection by applying digital low-pass filter [20] to the fluctuation of magnetization $M(t)$ and its Fourier component $M(t) * \cos(2\pi\omega t + \phi)$, where $\phi = \pi/2$ corresponds to the out-of-phase response

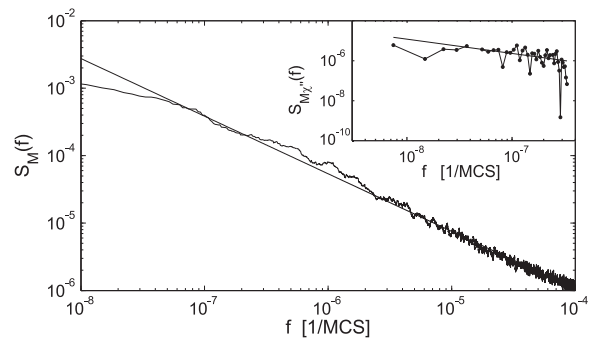


Figure 3: Normalized power spectra of the magnetization noise at $T = 0.05J_0$ of the lattice with $N = 1024$ sites. Linear fit yields $\alpha = 0.93$ for the noise exponent. Inset shows $S_{M\chi''}(f)$ at $T = 0.07J_0$, $\alpha = 0.72$.

χ'' . We take the field period to be relatively long, $\omega^{-1} = 3 \times 10^5$ MCS, for the system of $N = 1024$ spins. The resulting cross-correlator of the power spectra $S_{M\chi''}(f)/(S_M(f)S_{\chi''}(f))^{1/2}$, where $S_M(f) \equiv (1/M^2) \int_0^\infty dt e^{-2\pi f t i} M(0)M(t)$, is shown in Fig. 1. Temperature dependence of the cross-correlator of the noise amplitudes $\langle M\chi'' \rangle$ averaged over disorder is shown in the inset of Fig. 1; power spectra of the cross-correlator is shown in the inset of Fig. 3.

The noise spectrum and correlations observed in numerical simulations can be understood using the following qualitative picture. Long time scale dynamics responsible for the correlated noise in susceptibility and magnetization is generated by flipping parts of infinite percolation cluster and large clusters at its boundary. These flips are due to thermal activation, so the rate of the cluster to flip is $\Gamma \sim \exp(-V/T)$, where V is the maximal energy barrier encountered on the optimal spin flip path that flips the whole cluster.

Fractal geometry of clusters results in a logarithmic dependence of $V(s)$ on the cluster size. This unusual dependence was first seen in numerical work [23]; it can be understood theoretically by assuming that clusters that dominate the dynamics at relatively high frequencies are essentially random graphs [24]. Locally, any such structure can be viewed as a tree growing from the 'origin' that is connected by $m + 1$ bonds, of strength J , of which m connect to m sites of the first level that are connected by m bonds each to another m^2 sites of the second level and so on (see Fig. 4 for $m = 3$). The size of the cluster is $1 + (m^{L+1} - 1)/(m - 1)$ where L is the number of levels. Denote by $V_A = V_B = V_C = V$ the energy barrier to flip each of the subtrees A , B and C in Fig. 4. An optimal path to flip the whole tree involves flipping all subtrees in sequence which means that the maximum barrier will be $V + J$. This barrier is encountered when one of the subtrees, say A , have been flipped which requires keeping a bond from "0" to A in the high energy state. Subsequent flipping of B will therefore require $V + J$. This is the maximum barrier encountered because with the next

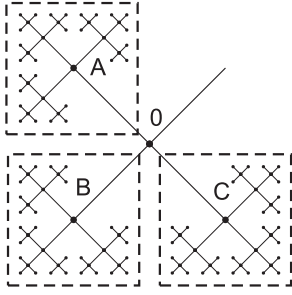


Figure 4: Sketch of the fractal spin structure illustrating the logarithmic dependence of the barrier with the cluster size.

step site "0" can be flipped reducing the barrier for C to $V + J$. In the case of $m > 3$ the maximum barrier will be $\approx V + m/2J$. This means that the energy barrier grows linearly with the number of levels whereas the size of the cluster is exponential. Therefore the resulting dependence of the energy barrier on the size of the cluster is logarithmic. One expects the logarithmic barrier scaling with size to remain valid also in the case of a random tree characterized by m and J fluctuating around average values $\langle m \rangle$ and $\langle J \rangle$, which is a more accurate model. In the case of percolation on the diluted Bethe lattice with fixed bond strength $J = \langle J \rangle$

$$V(s) \approx \frac{Z\langle J \rangle}{D} \ln s, \quad (2)$$

where D is the fractal dimension of the tree and the parameter Z is restricted by $1/\ln 2 \leq Z \leq 2/\ln 2$ [21]. A very similar argument based on the 'links' and 'blobs' structure of the cluster [22] supports the validity of scaling (2) for the whole cluster [24]. We expect this logarithmic dependence of the energy barriers on the cluster size should hold at least approximately for the system of spins interacting via exponential interaction (1) at percolation threshold with a similar value of Z .

The result (2) translates into the power law spectra

of the magnetization noise. At equilibrium the average autocorrelator of the magnetic moment $M_i(t)$ of the i -th cluster is given by $\overline{M_i(0)M_i(t)} \sim e^{-\Gamma t}$. Treating clusters as independent and approximating the cluster distribution by $n(V, s) \approx \delta(V - V(s))s^{-\tau}$, we get

$$S_M(f) \sim \int ds dV s^2 n(s, V) \frac{\Gamma(V)}{\Gamma^2(V) + f^2} \sim \frac{1}{f^{1+\zeta(T)}}, \quad (3)$$

$$\zeta(T) = \frac{T}{\langle J \rangle} \frac{D}{Z} (3 - \tau),$$

At percolation threshold the values of the parameters $\tau = 187/91$ and $D = 91/48$ are universal [25].

Out-of-phase response to the periodic magnetic field with relatively high frequency ω is dominated by small clusters with $\Gamma \sim \omega$ that do not equilibrate during one field period. These clusters experience a local quasistatic field created by all neighboring clusters that fluctuate with frequencies $f \ll \omega$. When one of these clusters flips the local field acting on the small cluster changes which results in the change in its response to the external field at frequency ω . This leads to the susceptibility fluctuations which are correlated with the magnetization fluctuations as observed in numerical simulations and in the experiment.

In conclusion, we have shown that the main results of recent experiments [11] on the flux noise are reproduced by the simplified model of a disordered ferromagnet with a very wide distribution of couplings between spins. In the simulations we assumed single spin flip dynamics which does not conserve local spin. Understanding the mechanism producing such dynamics or generalizing these results to other types of dynamics will be a subject of future work.

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circuit; Fourier analysis of the result allows one to extract long time $t\omega \gg 1$ behavior of susceptibility $\chi(t)$ of the surface spin system and quasistatic variations of the total flux at the same time scales. An alternative situation in which the inductance noise in SQUIDs is produced by dynamics of charged scattering centers was ruled out [26].

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